Polynomials

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Introduction

- You have studied algebraic expressions, their addition, multiplication and division in earlier classes.
- You may recall the algebraic identities:
  - $(x+y)^2=x^2+2xy+y^2$
  - $(x-y)^2=x^2-2xy+y^2$
  - $x^2-y^2=(x+y)(x-y)$
- In this chapter, we shall study with a particular type of algebraic expression, called polynomial and the terminology related to it.
- We shall also study the Remainder Theorem and Factor Theorem and their use in the factorisation of polynomials.
- We shall also study more algebraic identities and their use in factorisation and in evaluating some given expressions.

Polynomials in One Variable

- **Polynomial** is an expression of more than two algebraic terms, especially the sum of several terms that contain different powers of the same variable(s).
- A polynomial which consists of terms of different powers of the same variable in called **polynomial in one variable.** E.g. $3x^2+2x$ is a polynomial in variable x which is generally denoted as p(x).
- Each part of the polynomial is known as **term.** E.g. In polynomial $p(x)=5x^2-3x$, the terms are $5x^2$ and $3x$.
- A **coefficient** is a multiplicative factor in the terms of a polynomial. E.g. For the term $5x^2$, in the polynomial $p(x)= 5x^2-3x$, 5 is the coefficient of x^2.
- **Constant polynomials** are those which consists of only constants. E.g. 2, 3, -7 are constant polynomials.
- The constant polynomial 0 is also called **zero polynomial.**
- Polynomials with **one term** are called **monomials**, those with **two terms** are called **binomials** and those with **three terms** are called **trinomials.**

Polynomials in One Variable (Contd..)

- The highest power of the variable in a polynomial as the **degree of the polynomial.** Examples:
  - Degree of the polynomial $3x^7-4x^6+x+9$ is 7.
  - Degree of the polynomial $5y^6-4y^2-6$ is 6.
- A polynomial of degree one is called a **linear polynomial.** E.g. $3x+5$
- A polynomial of degree two is called a **quadratic polynomial.** E.g. $5y^2+4y$
- A polynomial of degree three is called a **cubic polynomial.** E.g. $2x^3+4x^2+6x+7$
- Thus, a polynomial in one variable x of degree n is an expression of the form:

  $a_nx^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0$

  where $a_0, a_1, a_2,\ldots, a_n$ are constants and $a\neq0$.

  In particular, if $a_0=a_1=a_2=a_3=\ldots=a_n=0$ (all the constants are zero), we get the zero polynomials

Zeroes of a Polynomial

- Zero of a polynomial p(x) is a number c such that p(c)=0.
- A non-zero constant polynomial has no zero. E.g. 5 is constant polynomial. It can also be written as $p(x)=5x^0$. Now, if we replace x by any number in $5x^0$ we still get 5.
- Every real number is a zero of the zero polynomial. This is because if we replace x by any number in 0x, we get 0 always.
- Let us consider an example. We will find the zeroes of polynomial x^2-2x.

  Now, let $p(x)=x^2-2x$.

  To get zeroes we put, p(x)=0, or $x^2-2x=0$, or x(x-2)=0, or x=0 or 2.

  Thus, 0 and 2 are the two zeroes of polynomial x^2-2x.
- Every linear polynomial has one and only one zero.
- A polynomial can have more than one zero.

Remainder Theorem

- In general, if p(x) and g(x) are two polynomials such that degree of p(x)≥degree of g(x) and g(x)≠0, then we can find polynomials q(x) and r(x) such that:

  $p(x)=g(x)q(x)+r(x)$,

  where r(x)=0 or degree of r(x)<degree of g(x). Here we say that p(x) divided by g(x), gives q(x) as quotient and r(x) as remainder.
- **Remainder Theorem**: Let p(x) be any polynomial of degree greater than or equal to one and let a be any real number. If p(x) is divided by the linear polynomial x–a, then the remainder is p(a).
- E.g. Find the remainder when $x^4+x^3-2x^2+x+1$ is divided by x–1.

  Here, applying remainder theorem, a=1.

  So, Remainder=$p(a)=p(1)=(1)^4+(1)^3-2(1)^2+1+1$

  $=2$

  Thus, 2 is the remainder when $x^4+x^3-2x^2+x+1$ is divided by x–1.

Factorisation of Polynomials

- **Factor Theorem**: If p(x) is a polynomial of degree $n \geq 1$ and a is any real number, then (i) x−a is a factor of p(x), if p(a)=0, and (ii) p(a)=0, if x−a is factor of p(x).
  Proof: By the remainder theorem, p(x)=(x−a)q(x)+p(a).
  (i) If p(a)=0, then p(x)=(x−a)q(x), which shows that x−a is a factor of p(x).
  (ii) Since x−a is a factor of p(x), p(x)=(x−a)g(x) for some polynomial g(x).
  In this case, p(a)=(a−a)g(a)=0.

- Factorisation of the polynomial ax^2+bx+c **by splitting the middle term** is as follows:
  Let the factors of ax^2+bx+c be $(px+q)$ and $(rx+s)$. Then,
  $ax^2+bx+c=(px+q)(rx+s)=pr\ x^2+(ps+qr)x+qs$
  Now comparing the coefficients, we get $a=pr$, $b=ps+qr$ and $c=qs$.
  This shows that b is the sum of two numbers ps and qr, whose product is $(ps)(qr)=(pr)(qs)=ac$.

Algebraic Identities

- **Algebraic Identities:**

  Identity I : $(x+y)^2=x^2+2xy+y^2$

  Identity II : $(x-y)^2=x^2-2xy+y^2$

  Identity III : $x^2-y^2=(x+y)(x-y)$

  Identity IV : $(x+a)(x+b)=x^2+(a+b)x+ab$

  Identity V : $(x+y+z)^2=x^2+y^2+z^2+2xy+2yz+2zx$

  Identity VI : $(x+y)^3=x^3+y^3+3xy(x+y)$

  Identity VII : $(x-y)^3=x^3-y^3-3xy(x-y)$

  $=x^3-3x^2y+3xy^2-y^3$

  Identity VIII: $x^3+y^3+z^3-3xyz=(x+y+z)(x2+y2+z2-xy-yz-zx)$

Summary

- A polynomial p(x) in one variable x is an algebraic expression in x of the form
  $P(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a^2x^2+a_1x+a_0$
  where a_0, a_1, a_2,…, a_n are constants and $a\neq0$.
  a_0, a_1, a_2,…, a_n are respectively the coefficients of $x^0,x,x^2,\ldots,x^n$ and n is called the degree of the polynomial. Each of $a_nx_n,a_{n-1}x_{n-1},\ldots,a_0$, with $a_n\neq0$, is called the term of the polynomial p(x).
- A polynomial of one term is called a monomial.
- A polynomial of two terms is called a binomial.
- A polynomial of three terms is called a trinomial.
- A polynomial of degree one is called a linear polynomial.
- A polynomial of degree two is called a quadratic polynomial.
- A polynomial of degree three is called a cubic polynomial.
- A real number ‘a’ is a zero of a polynomial p(x) if p(a)=0. In this case, a is also called a root of the equation p(x)=0

Summary (Contd..)

- Every linear polynomial in one variable has a unique zero, a non-zero constant polynomial has no zero, and every real number is a zero of the zero polynomial.
- **Remainder Theorem** : If p(x) is any polynomial of degree greater than or equal to 1 and p(x) is divided by the linear polynomial x – a, then the remainder is p(a).
- **Factor Theorem** : x – a is a factor of the polynomial p(x), if p(a) = 0. Also, if x – a is a factor of p(x), then p(a) = 0.
- $(x+y+z)^2=x^2+y^2+z^2+2xy+2yz+2zx$
- $(x+y)^3=x^3+y^3+3xy(x+y)$
- $(x-y)^3=x^3-y^3-3xy(x-y)$
- $x^3+y^3+z^3-3xyz=(x+y+z)(x^2+y^2+z^2-xy-yz-zx)$

THANK YOU